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ON 4-DIM DUCK SOLUTIONS WITH RELATIVE STABILITY

KIYOYUKI TCHIZAWA

Directorate. KanriKogaku Kenkyusho Ltd.
(Institute of Administration Engineering)

ABSTRACT. This paper gives the existence of a relatively stable duck solution in a slow-fast system in R^{2+2} with an invariant manifold. It has a 4-dimensional duck solution having a relatively stable region when there exist the invariant manifold near the pseudo singular node point.

1.INTRODUCTION

In a previous paper [5], H.Nishino, H.Miki and the author have constructed 2-dimensional duck solutions in Goodwin's economic model modifying the effective function. In another one [6], we got 4-dimensional ducks in a trading economic model using two symmetric Goodwin's models. These results lead us a new point of view to analyze the stability for the ducks. In the R^{2+2} slow-fast system with an invariant manifold, we first assume that this manifold describing limit cycle has a duck solution in a projected R^2 space. If there exists pseudo singular node point near the invariant manifold, the system has a duck solution with a relatively stable region in R^4. This fact gives a global behavior in R^4, because it satisfies the condition including the invariant manifold at around the pseudo singular point. In other words, we can observe a center manifold for the slow-fast system in R^4.

2.SLOW–FAST SYSTEM IN R^2

In this section, we shall review some results in Zvonkin and Shubin[9]. Let us consider the following system of differential equations

$$\begin{aligned} \epsilon dx/dt &= w - f(x), \\ dw/dt &= a - x, \end{aligned} \tag{2.1}$$

where f is defined in R^1 and ϵ is infinitesimal. For the system (2.1), the graph $w = f(x)$ is called the *slow curve.* We consider the extremum point x_0 that separates the attracting part and the repelling part.

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Definition2.1. A solution $(x(t), w(t))$ of the system (2.1) is called a *duck solution* if there exist standard numbers t_1, t_0, t_2 $(t_1 < t_0 < t_2)$ such that

(1)$^*[x(t_0)] = x_0$, where $^*[X]$ denotes the standard part of X, (2) for $t \in (t_1, t_0)$ the segment of the trajectory $(x(t), w(t))$ is infinitesimally close to the attracting part of the slow curve, (3) for $t \in (t_0, t_2)$, it is infinitesimally close to the repelling part of the slow curve, and (4) the attracting and repelling parts of the trajectory are not infinitesimal.

We give a necessary condition for the existence of a duck solution close to the extremum point x_0 of $f(x)$.

Proposition2.1. If there is a duck solution of the system (2.1) close to the extremum point x_0, then $a \approx x_0$.

We finally obtain the following proposition concerning the existence of duck solutions.

Proposition2.2. Suppose that f has a nondegenerate extremum point x_0, that is, $f'(x_0) = 0$ and $f''(x_0) > 0$. Then there are the corresponding values of the parameter a satisfying Proposition2.1 for which there exist duck solutions in the system (2.1).

3.SLOW-FAST SYSTEM IN R^4

Now, let us consider a slow-fast system(3.1):

$$
\begin{aligned}
\epsilon dx_1/dt &= h_1(x_1, x_2, y_1, y_2, \epsilon), \\
\epsilon dx_2/dt &= h_2(x_1, x_2, y_1, y_2, \epsilon), \\
dy_1/dt &= f_1(x_1, x_2, y_1, y_2, \epsilon), \\
dy_2/dt &= f_2(x_1, x_2, y_1, y_2, \epsilon),
\end{aligned} \tag{3.1}
$$

where $f = (f_1, f_2)$ and $h = (h_1, h_2)$ are standard defined on $R^4 \times R^1$ and ϵ is infinitesimal.

First, we assume the following condition $(A1)$ to get an explicit solution.

$(A1)$ f is of class C^1 and h is of class C^2.

Furthermore, we assume that the system(3.1) satisfies the following generic conditions $(A2) - (A5)$:

$(A2)$ The set $S_2 = \{(x, y) \in R^4 | h(x, y, 0) = 0\}$ is a 2-dimensional differentiable manifold and the set S_2 intersects the set $T_2 = \{(x, y) \in R^4 | det[\partial h(x, y, 0)/\partial x] = 0\}$, which is a 3-dimensional differentiable manifold, transversely so that the generalized pli set $GPL = \{(x, y) \in S_2 \cap T_2\}$ is a 1-dimensional differentiable manifold.

$(A3)$ The value of f is nonzero at any point $p \in GPL$.

$(A4)$ The $rank[\partial h(x, y, 0)/\partial x] = 2$ for any $(x, y) \in S_2 \backslash GPL$, and the $rank[\partial h(x, y, 0)/\partial y] = 2$ for any $(x, y) \in S_2$. Then, the surface S_2 can be expressed as $y = \varphi(x)$ in the neighborhood of GPL.

Assume $y = \varphi(x)$. On the set S_2, differentiating both sides of $h(x, \varphi(x), 0) = 0$ with respect to x,

$$[h_x] + [h_y]D\varphi = 0, \tag{3.2}$$

where $D\varphi$ is a derivative with respect to x, thus the following is established:

$$D\varphi(x) = -[h_y]^{-1}[h_x]. \tag{3.3}$$

On the other hand,

$$dy/dt = D\varphi(x)dx/dt, \tag{3.4}$$

because of $y = \varphi(x)$. We can reduce the slow system to the following:

$$D\varphi(x)dx/dt = f(x, \varphi(x)). \tag{3.5}$$

Using(3.3), the system (3.5) is described by

$$[h_x]dx/dt = -[h_y]f(x, \varphi(x)). \tag{3.6}$$

Put $A = [h_x] = [h_{ij}]$ simply, then

$$dx/dt = -B[h_y]f(x, \varphi(x)), \tag{3.7}$$

where B is a cofactor matrix of A, that is, $B = [A_{ji}]$. A_{ij} is a *cofactor* of h_{ij}.

($A5$)The system(3.7) is the time scaled reduced system projected into R^2. Again, we assume the set $T_2 = \{(x, y) \in R^4 | detA = 0\} \neq \phi$. All the singular points of the system(3.7) are nondegenerate, that is, the matrix induced from the linearized system of (3.7) at a singular point has distinct nonzero eigenvalues.

Remark. All these points are contained in the set $GPS = \{(x, y) \in GPL | detA = 0\}$, which is called the set of *generalized pseudo singular points.*

As this approach transforms the original system to the time scaled reduced system directly, it is called a *direct method.*

Definition3.1. Let $p \in GPS$ and μ_1, μ_2 be two eigenvalues of the matrix associated with the linearized system of (3.7) at $p \in R^4$. The point p is called *generalized pseudo singular saddle* if $\mu_1 < 0 < \mu_2$ and called *generalized pseudo singular node* if $\mu_1 < \mu_2 < 0$ or $\mu_1 > \mu_2 > 0$. It is called *generalized pseudo singular focus* if they are compex conjugate.

Now, we have to give a description on the definition of the duck solution in R^4 along the direct method.

Definition3.2. Let a point p be in GPS. If a trajectory follows first the attractive surface before this point and the saddle one at the point p, and then it goes along the slow manifold, which is not infinitesimal, it is called a *duck solution in R^4*.

Furthermore, we assume that the following.

($A6$) The invariant manifold $Inv(h(x, y))$ lying near the GPS has 2-dimension in R^4. It intersects GPL transversely.

4.Local models

In this section, we shall give the following two theorems through a local model in R^{2+2}. See [7].

Theorem4.1. Let $0 \in GPS$ be saddle or node. If the matrix $[h_x(0, \phi(0))]$ has one zero eigenvalue and the other one has negative with a local model satisfying the conditions: $(1)\partial h_1(0)/\partial x_2 = 0, \partial h_2(0)/\partial x_2 = 0$, $(2) f_1(0) \not\simeq 0, f_2(0) \not\simeq 0$, there exists a duck solution in R^4.

(Proof) As only one of the eigenvalues of the matrix $[h_x(x, \phi(x))]$ on the slow manifold takes zero on GPS, the assumptions $(A2)$, $(A4)$ ensure that two eigenvalues of $[h_x(x, \phi(x))]$ are negative in the fast vector field before GPS. They are maybe it is meant negative, respectively positive after GPS. When each coefficient on GPS is limitted, a local model shows a precise structure as an approximation of the original system. Then, the property on GPS reflects direc tly the whole system. It can be shown that the time scaled reduced system ($\epsilon = 0$) is an apprximated one with a singular solution of the whole system ($\epsilon \neq 0$), because the corresponding solutions are very close each other under the only two conditions. Therefore, we can cnclude that there exists a duck solution.

Let $0 \in GPS$ be saddle or node. When changing the variables correspond to microscopes ($\alpha \simeq 0$): $x_1 = \alpha^p u_1, x_2 = \alpha^q u_2, y_1 = \alpha^r v_1, y_2 = \alpha^s v_2$, $p, q, r, s \in N$, the original system is reduced to the system with variables u_1, u_2, v_1, v_2. Then there exist local models which describe the 4-dimensional duck solutions.

Theorem4.2. If the system has a square-linear solution in a local model, for any $p, q, r, s \in N$, there exist essentially two local models describing the explicit duck solutions, .

(Proof)

In the case $p = 2$, $q = 1$, $r = 2$, $s = 2$, changing variables:

$$x_1 = \alpha^2 u_1, x_2 = \alpha u_2, y_1 = \alpha^2 v_1, y_2 = \alpha^2 v_2, \tag{4.1}$$

we reduce the system as well in (4.2) as well in (4.3).

$$\begin{aligned} \epsilon du_1/dt &= h_1(\alpha^2 u_1, \alpha u_2, \alpha^2 v_1, \alpha^2 v_2, \epsilon)/\alpha^2, \\ \epsilon du_2/dt &= h_2(\alpha^2 u_1, \alpha u_2, \alpha^2 v_1, \alpha^2 v_2, \epsilon)/\alpha, \\ dv_1/dt &= f_1(\alpha^2 u_1, \alpha u_2, \alpha^2 v_1, \alpha^2 = 20 v_2, \epsilon)/\alpha^2, \\ dv_2/dt &= f_2(\alpha^2 u_1, \alpha u_2, \alpha^2 v_1, \alpha^2 v_2, \epsilon)/\alpha^2. \end{aligned} \tag{4.2}$$

Multiplying the right hand side of the system(3.2) by α^2,

$$\begin{aligned} (\epsilon/\alpha^2) du_1/dt &= h_1(\alpha^2 u_1, \alpha u_2, \alpha^2 v_1, \alpha^2 v_2, \epsilon)/\alpha^2 \\ (\epsilon/\alpha^2) du_2/dt &= h_2(\alpha^2 u_1, \alpha u_2, \alpha^2 v_1, \alpha^2 v_2, \epsilon)/\alpha, \\ dv_1/dt &= f_1(\alpha^2 u_1, \alpha u_2, \alpha^2 v_1, \alpha^2 v_2, \epsilon), \\ dv_2/dt &= f_2(\alpha^2 u_1, \alpha u_2, \alpha^2 v_1, \alpha^2 v_2, \epsilon). \end{aligned} \tag{4.3}$$

In fact, doing time scaling $t = \alpha^2 \tau$, then $dt = \alpha^2 d\tau$. It is easily shown that the formula(4.3) is equivalent to (4.2).

By using the assumptions $(A1)$ and $(A4)$, we construct a local model under the most simple conditions:

$$\begin{aligned} &(1)\partial h_1(0)/\partial x_2 = 0, \partial h_2(0)/\partial x_2 = 0, \\ &(2) f_1(0) \not\simeq 0, f_2(0) \not\simeq 0. \end{aligned} \tag{4.4}$$

Putting ϵ/α^2 infinitesimal to ϵ simply, the local model reduced from the system(3.1) is obtained.

$$\begin{aligned} \epsilon du_1/dt &= Au_1 + Bv_1 + Cv_2 + Du_2^2/2 + L(\epsilon), \\ \epsilon du_2/dt &= Eu_2 + L(\epsilon), \\ dv_1/dt &= f_1(0) + L(\epsilon), \\ dv_2/dt &= f_2(0) + L(\epsilon), \end{aligned} \tag{4.5}$$

where $A = \partial h_1(0)/\partial x_1$, $B = \partial h_1(0)/\partial y_1$, $C = \partial h_1(0)/\partial y_2$, $D = \partial^2 h_1(0)/\partial x_2^2$, $E = \partial h_2(0)/\partial x_2$.

Note that the conditions $A = \partial h_1(0)/\partial x_1 < 0$ and $E = \partial h_2(0)/\partial x_2 = 0$ imply that $0 \in GPS$ is saddle. See Definition3.3. The corresponding solutions in the local model are as follows: when $\epsilon = 0$,

$$\begin{aligned} &u_1 = -(Bf_1(0) + Cf_2(0))t/A - Dt^2/(2A), u_2 = t, \\ &v_1 = f_1(0)t, v_2 = f_2(0)t, \end{aligned} \tag{4.6}$$

when $\epsilon \neq 0$,

$$\begin{aligned} &u_1 = -(Bf_1(0) + Cf_2(0))t/A - Dt^2/(2A) + L(\epsilon), u_2 = t + L(\epsilon), \\ &v_1 = f_1(0)t + L(\epsilon), v_2 = f_2(0)t + L(\epsilon). \end{aligned} \tag{4.7}$$

In the case $p = 2$, $q = 1$, $r = 3$, $s = 2$, changing variables:

$$x_1 = \alpha^2 u_1, x_2 = \alpha u_2, y_1 = \alpha^3 v_1, y_2 = \alpha^2 v_2, \tag{4.8}$$

we construct a local model under the conditions:

$$\begin{aligned} &(1)\partial h_1(0)/\partial x_2 = 0, \partial h_2(0)/\partial x_2 = 0, \\ &(2) f_1(0) = 0, f_2(0) \not\simeq 0. \end{aligned} \tag{4.9}$$

The corresponding local model is

$$\begin{aligned} \epsilon du_1/dt &= Au_1 + Bv_2 + Cu_2^2/2 + L(\epsilon), \\ \epsilon du_2/dt &= Du_2 + L(\epsilon), \\ dv_1/dt &= Eu_2 + L(\epsilon), \\ dv_2/dt &= f_2(0) + L(\epsilon), \end{aligned} \tag{4.10}$$

where $A = \partial h_1(0)/\partial x_1$, $B = \partial h_1(0)/\partial y_2$, $C = \partial^2 h_1(0)/\partial x_2^2$, $D = \partial h_2(0)/\partial x_2$, $E = \partial f_1(0)/\partial x_2$.

Notice that we assume again that $A < 0$ and $D = 0$, because the fast vector field has one zero eigenvalue and the other one is negative.. The corresponding solutions in the local model are as follows: when $\epsilon = 0$,

$$\begin{aligned} &u_1 = -Bf_2(0)t/A - Ct^2/(2A), u_2 = t, \\ &v_1 = Et^2/2, v_2 = f_2(0)t, \end{aligned} \tag{4.11}$$

when $\epsilon \neq 0$,

$$u_1 = -Bf_2(0)t/A - Ct^2/(2A) + L(\epsilon), u_2 = t + L(\epsilon),$$
$$v_1 = Et^2/2 + L(\epsilon), v_2 = f_2(0)t + L(\epsilon). \tag{4.12}$$

In another case, it is impossible to get an explicit solution with a square-linear one but a cubic-linear (or much higher order) one.

In this approach, an invertible affine transformation must be needed for a general point $p \in GPS$, because the conditions (4.4), (4.9) are assumed at only $0 \in GPS$. These conditions may not be satisfied at the general pseudo singular point. We have to change the coordinates from the point p to 0. Notice that we do not know if the corresponding affine transformation keeps the conditions(4.4). In many cases, however, it is feasible.

Remark. It is easily to find that any solutions (u_1, u_2, v_1, v_2) at the same time t in (4.6) and (4.7) are very near. This fact implies that the time scaled reduced system is an approximated one.

5.Relative Stability

In what follows we shall show how to construct a 4-dimensional duck with relative stability[9]. In the system (3.1), we assume that the following: when $(1)h_1 = h_2, f_1 = f_2$, and $(2)x_1 = x_2, y_1 = y_2$, are satisfied, the system has a 2-dimensional duck on a projected space (x_1, y_1). Notice that the corresponding invariant manifold $Inv(h(x,y))$ is a limit cycle including the duck. In this state, the manifold intersects GPL transversely.

Definition 5.1. If the system (3.1) satisfies the above condition (1), it is said to be symmetric.

Definition5.2. Let a compact set M and a set V be in R^4. The set M is said to be stable, relatively to the set V, if given an $\epsilon > 0$ there exists $\delta > 0$ such that $\gamma(U(M,\delta) \cap V, t) \subset U(M,\epsilon)$ for any $t \in R$, where $U(M,\epsilon)$ denotes ϵ-neighborhood of the set M, and $\gamma((x,y),t)$ denotes the trajectory.

Theorem5.1. Let the system(3.1) be symmetric and have a 2-dimensional duck. If there exists a pseudo singular node point in the ϵ^3-neighborhood of the invariant set $Inv(h(x,y))$, then there exists a 4-dimensional duck with relative stability.

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References

1. E.Benoit, *Canards et enlacements*, Publ. Math. IHES **72** (1990), 63–91.
2. E.Benoit, *Canards en un point pseudo-singulier noeud*, Bulletin de la SMF (1999), 2–12.
3. S.A. Campbell, M.Waite, *Multistability in Coupled Fitzhugh-Nagumo Oscillators*, Nonlinear Analysis **47** (2000), 1093–1104.
4. K.Tchizawa, S.A.Campbell, *On winding duck solutions in* R^4, Proceedings of Neural, Parallel, and Scientific Computations **2** (2002), 315–318.
5. K.Tchizawa, H.Miki, H.Nishino, *On the existence of a duck solution in Goodwin's nonlinear business cycle model*, Nonlinear Analysis **63** (2005), e2553–e2558.
6. H.Miki, K.Tchizawa, H.Nishino, *On the possible occurrence of duck solutions in domestic and two-region business cycle models*, preprint.
7. K.Tchizawa, *On a direct method for proving existence of 4-dim duck solutions*, preprint.
8. A.K. Zvonkin and M.A. Shubin, *Non-standard analysis and singular perturbations of ordinary differential equations*, Russian Math. Surveys **39** (1984), 69-131.
9. N.P.Bhatia and G.P.Szego, *Stability Theory of Dynamical Systems*, Springer **161** (1970).

2-2-2 Sotokanda Chiyoda-Ku, Tokyo, 101-0021, Japan